NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

COPY NO.

IMM-NYU 210
JUNE 1954

IMM 210

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

On the Theory of Flood Waves in Rivers

GEORGE K. MORIKAWA

PREPARED UNDER
CONTRACT NO. DA-33-017-Eng-223
U. S. ARMY ENGINEER CORPS
OHIO RIVER DIVISION

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

IMM-NYU 210
June 1954

NEW YORK UNIVERSITY
Institute of Mathematical Sciences

ON THE THEORY OF FLOOD WAVES IN RIVERS

George K. Morikawa

Contract DA-33-0i7-Eng-223

Prepared under the sponsorship of
U. S. Army Engineer Corps, Ohio River Division

New York, 1954

On the Theory of Flood Waves in Rivers

George K. Morikawa

1. Introduction

Flood waves, in rivers with small bottom slope such that subcritical flow only is encountered, are known to progress downstream with a velocity substantially less than the propagation velocity of small amplitude gravity waves, (i.e. the downstream characteristic velocity). Such is the case, for example, in the Ohio River where a goodly amount of experimental data during flood conditions have been obtained over a period of many years by Army engineers. On the basis of such data hydraulic engineers have devised a relatively satisfactory engineering procedure for flood prediction and control (called flood routing); a description is given by Gilcrest[1]. However, the recent advent of high speed computing techniques and equipment has stimulated renewed interest and effort in improving present procedures by directly attacking the basic non-linear differential equations governing flow in open channels (cf. NYU reports by Stoker [2] and Isaacson, Stoker and Troesch [3]). The present report is concerned with a study of the significance of the propagation velocity of flood waves and its relationship to the propagation velocities which characterize the basic differential equation system.

The basic differential equations for flow in open channels are well-established. In the present investigation we restrict ourselves to the simplest possible form of the equations, namely, flow in a uniform channel over a bottom of constant slope. Then the flow system is completely described by two equations of continuity and momentum for the channel depth h and the flow velocity v as functions of the time t and the streamwise distance x.

(1a) $h_t + (hv)_x = 0$

(1b) $v_t + vv_x + gh_x + \frac{Kv^2}{h} - gS = 0$

where g is the acceleration of gravity, K is the turbulence resistance coefficient (Chézy) and S is the constant bottom slope. Subscripts denote differentiation. The turbulence resistance term Kv^2/h can be written in an alternate form $gv^2/_{\gamma c^2}$ where $c = (gh)^{\frac{1}{2}}$ is the signal velocity and $\gamma \equiv 1/K$ corresponds to a viscosity coefficient. Thus large K, corresponding to high turbulence resistance, implies small γ corresponding to small viscosity. The dimensionless velocity ratio v/c is called the Froude number, based on a characteristic length which, in open channel flow, is the depth. In the analogy with compressible gas dynamics, v/c is called the Mach number. Although this analogy exists we emphasize that from a physical viewpoint the Froude number and Mach number are distinct; the former is the ration of inertia to gravity forces and the latter is the ratio of a velocity of flow to the signal velocity of small disturbances. (1b) can be regarded as an approximate momentum equation valid under the shallow water approximation and obtainable by a formal mathematical procedure from Euler's equations of motion, e.g. as described by Friedrichs [4] and Keller [5], with the addition of a correction term for turbulence resistance. However, we prefer to regard (1b) as directly derivable from physical considerations, e.g. as obtained by Stoker [2], where the dependent variables (h,v) are considered to be average values of channel depth and flow velocity. Then the turbulence resistance term can be reasonably justified on the basis of Prandtl's

"mixing length theory" of turbulence [6]. The implication of this viewpoint is reserved for the discussion in Section 3. In application to actual rivers, the basic equations corresponding to (1) include additional terms due to variable channel cross-section, bottom slope and overbank flow. Also other forms of the turbulence resistance term, which seems to depend primarily on the shape of the channel cross-section, are used by hydraulic engineers. We regard these factors as correction terms which will modify but not substantially change our conclusions reached on the basis of our study of (1).

The basic differential equations (1) are of hyperbolic type similar to those of compressible gas dynamics, a subject which has been of considerable interest to applied mathematicians, physicists and engineers in recent years. The usual method of treating this type of equations is by the method of characteristics, e.g. as shown in [7] and, for explicit solutions of initial and boundary value problems for these equations, by finite difference methods, e.g. as discussed in the NYU reports [2] and [3]. However because of the presence of the turbulence resistance term in (1) we find that, in addition to the two characteristic velocities $(v \pm c)$, a propagation velocity U is defined where $v \leq U < (v+c)$. U is the velocity of the known steady progressing wave solution (cf. Section 2) which is the basis of the usual flood routing procedures (cf. [3]). Previous investigators studying the theory of flood waves have noted the significance of U in their analyses. Deymié [8] has studied* the linearized form (perturbation on the constant steady flow solution (h_o,v_o)) of (1), resulting in a form of the "telegraph equation" for both the perturbed channel depth and flow velocity. He considered a problem

---

* G. B. Whitham has independently studied this linearization by a somewhat different analysis but with substantially similar results in an unpublished report at NYU.

with radiation type boundary conditions appropriate for the study of flood waves: given zero initial conditions for time $t = 0$ and prescribed channel depth or flow velocity as a function of time at a fixed point $x = 0$ for $t > 0$, to determine the behavior of these perturbed quantities downstream $(x > 0)$ and for $t > 0$. His analysis shows that the initial disturbance travels downstream with the linearized characteristic velocity $(v_0 + c_0)$ but that the main or maximum disturbance, which initially travels with this velocity, moves downstream with a velocity which asymptotically (large t) approaches the propagation velocity U of the steady progressing wave. All disturbances are completely damped eventually. Although this linearized theory yields valuable information on the behavior of certain types of disturbance waves, it is not completely satisfactory in the sense that solutions of the type of the steady progressing wave are not included. Recently Hayashi [9] has presented an approximate solution of the initial value problem for (1) by an iteration procedure, but it has been pointed out by A. Troesch* that this solution leads to "breaking", that is, bores (or shocks) eventually develop, even for disturbances like the steady progressing wave. In this report these questions are clarified. The practical applicability of our study to actual flood waves will be included in a future report by A. Troesch.

---

* Private communication.

2. Approximate Representation of Steady Progressing Wave and Stretching Transformation.

Since one of our primary interests is in obtaining approximate time-dependent solutions (waves) which travel with the velocity of the steady progressing wave*, and which will include the SPW solution, we transform (1) to a coordinate system moving downstream with this velocity U, namely, $(x,t) \longrightarrow (\sigma,t)$ where $\sigma = x - Ut$. Then in the new coordinate system, (1) becomes

$$h_t + [h(v-U)]_\sigma = 0 \tag{2a}$$

$$v_t + (v-U)v_\sigma + gh_\sigma + \frac{Kv^2}{h} - gS = 0 \tag{2b}$$

Setting $\frac{\partial}{\partial t} = 0$ in (2) we obtain the system of ordinary differential equations satisfied by the SPW solution denoted by $\bar{h}(\sigma)$ and $\bar{v}(\sigma)$. From (2a) we determine

$$(\bar{v} - U) = \frac{h_o(v_o-U)}{\bar{h}} \tag{3a}$$

which we substitute into (2b) yielding a first order ordinary differential equation for the wave profile $\bar{h}(\sigma)$

$$[c_o^2(\bar{h}/h_o)^3 - v_o^2(1 - U/v_o)^2]\bar{h}_\sigma + Kv_o^2\left\{[1 + (\bar{h}/h_o - 1)U/v_o]^2 - (\bar{h}/h_o)^3\right\} = 0 \tag{3b}$$

---

[*] Hereafter, "steady progressing wave" is denoted by "SPW" and refers to the uniquely determined, bounded, non-trivial solution of the differential equation (3).

where $c_o = (gh_o)^{\frac{1}{2}}$, $h(\infty) = h_o$ and $v(\infty) = v_o$. The propagation velocity U is found directly from (2a) for $\frac{\partial}{\partial t} = 0$:

$$(4) \qquad U/v_o = \frac{(h_1/h_o)^{\frac{3}{2}} - 1}{(h_1/h_o - 1)}$$

where $h(-\infty) = h_1 > h_o$ and

$$(5) \qquad \frac{v_1^2}{h_1} = \frac{v_o^2}{h_o}$$

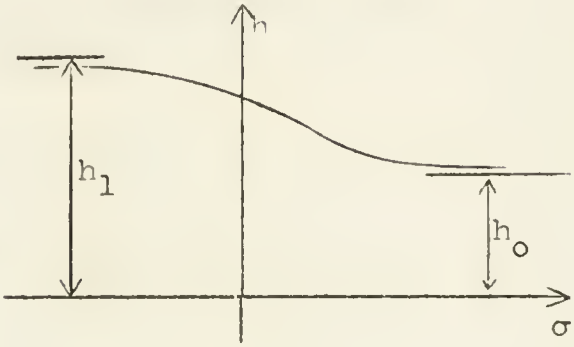


Fig. 1

The shape of the SPW profile is shown in Fig. 1. We note here that the constant solution h_o satisfies (3) and the constant solution v_o satisfies the corresponding differential equation for $\overline{v}(\sigma)$, that is, (h_o, v_o) is the steady flow solution which satisfies both (1) and (2) with the condition

$$(6) \qquad \frac{Kv_o^2}{h_o} - gS = 0$$

Both the steady flow and SPW solutions are discussed in greater detail in Appendix II of [3].

In looking for approximate time-dependent solutions related to the SPW solution an obvious generalization of the ordinary linearization of Deymié [8] is to carry out a perturbation expansion with respect to a small parameter ε, say, on the known steady progressing wave solution. Putting

$$h(\sigma,t) = \bar{h}(\sigma) + \varepsilon h^{(1)}(\sigma,t) + \varepsilon^2 h^{(2)}(\sigma,t) + \dots$$

$$v(\sigma,t) = \bar{v}(\sigma) + \varepsilon v^{(1)}(\sigma,t) + \varepsilon^2 v^{(2)}(\sigma,t) + \dots$$

in (2) and equating the coefficients of the first power in ε, we obtain the first order system

$$\text{(7a)} \quad h_t^{(1)} + [(\bar{v} - U)h^{(1)} + \bar{h}v^{(1)}]_\sigma = 0$$

$$\text{(7b)} \quad v_t^{(1)} + [(\bar{v} - U)v^{(1)} + gh^{(1)}]_\sigma + \frac{K\bar{v}^2}{\bar{h}}\left(\frac{2v^{(1)}}{\bar{v}} - \frac{h^{(1)}}{\bar{h}}\right) = 0$$

which is a linear system for $(h^{(1)},v^{(1)})$ with non-constant coefficients depending on the known solution $(\bar{h},\bar{v})$. The particular case $(\bar{h},\bar{v}) = (h_o,v_o)$ is Deymié's first order linearized system. However, explicit solutions to (7) are difficult to obtain so we pursue another approach.

Returning to the SPW equations (3), (4) and (5), we look for an expansion procedure which will approximate the SPW solution for small values of the appropriate parameter $\lambda = (h_1 - h_o)/h_o \ll 1$, i.e., for small values of the wave height $(h_1 - h_o)$ with respect to the depth h_o. So we let

$$\text{(8)} \quad h/h_o = (1 + \tilde{h})$$

where $h_1/h_o = (1 + \lambda)$ and $0 \leq \tilde{h} \leq \lambda$. (4) can be expanded as a function of λ only yielding

$$\text{(9a)} \quad U = U_o + \lambda U_1 + \lambda^2 U_2 + \dots$$

or specifically for the steady progressing wave

$$(9b) \qquad U/v_o = \frac{3}{2}\left(1 + \frac{\lambda}{4} - \frac{\lambda^2}{24} + \dots\right)$$

Putting (8) and (9) in (3) and retaining only "first order" terms in $\tilde{h}$ and λ gives

$$(10) \qquad \tilde{h}_\sigma = 0$$

which yields the trivial constant solution. However, we note that the first order non-derivative terms are identically zero forcing us to retain the second order non-derivative terms. Then we obtain

$$(11a\ ) \qquad \tilde{h}_\sigma + A\tilde{h}(\lambda - \tilde{h}) = 0$$

where

$$(11b) \qquad A = (3Kv_o^2/4h_o)/(c_o^2 - v_o^2/4)$$

For $v_o/c_o < 2$, the solution of (11) is

$$(12) \qquad \tilde{h}(\sigma) = \frac{\lambda}{2}\left(1 - \tanh \frac{A\lambda\sigma}{2}\right)$$

which satisfies the required properties of the lowest order approximate solution representing the steady progressing wave and shows that the derivative term $\tilde{h}_\sigma$ in (11) is actually of second order in λ and that this lowest order differential equation satisfied by the SPW solution is non-linear. (12) shows that $\tilde{h}$ is a function of $(\lambda\sigma)$ and gives us the motivation of applying a stretching transformation $\omega = \lambda\sigma$, say, (cf. Refs. [4] and [5]) with an expansion procedure

with respect to the parameter λ in order to obtain approximate solutions of (2) which will include the SPW solution. It is now clear why the ordinary linearization procedure of Deymié does not include the SPW solution. In the following two sections we shall see that simultaneous stretching transformations on both σ and t yield asymptotic representations of the basic differential equation system (2) and consequently asymptotic solutions of (2).

3. Asymptotic Representation for (λ,λ^2)* Stretching Transformation.

A priori it is not clear from the basic differential equation system (2) what type of stretching on the time variable will yield physically reasonable asymptotic representations. It appears that we must depend rather heavily on our knowledge of the order of magnitude of various terms in (2) based on physical considerations, being especially mindful of the appropriateness of (2) in reasonably describing our actual physical system. Thus, the stretching transformation

$$\omega = \lambda\sigma \,, \quad \tau = \lambda^2 t \tag{13}$$

transforms (2) into

$$\lambda h_{\tau} + [h(v - U)]_{\omega} = 0 \tag{14a}$$

$$\lambda\left\{\lambda v_{\tau} + [(v - U)v_{\omega} + gh_{\omega}]\right\} + \frac{Kv^2}{h} - gS = 0 \tag{14b}$$

---

* This notation is used to describe a stretching transformation on $(\sigma,t) \longrightarrow (\omega,\tau)$ where $\omega = \lambda\sigma$ and $\tau = \lambda^2 t$.

Hence the (λ, λ^2) transformation defined by (13) implies that in (2) the time derivative terms are of lower order of magnitude (or higher order in λ) than the space derivative terms which are in turn of lower order of magnitude than the non-derivative terms. Historically, such arguments on orders of magnitude of various terms in the Navier-Stokes equations of motion based on physical considerations led to Prandtl's concept of the boundary layer. For $\lambda = 0$ in (14)

$$(15a) \qquad [h(v - U)]_{\omega} = 0$$

$$(15b) \qquad \frac{Kv^2}{h} - gS = 0$$

which is satisfied by the steady flow solution $(h,v) = (h_o, v_o)$, constant, where h_o and v_o are related by (6). Therefore we make perturbation expansions with respect to the parameter λ of (h,v) on the constant solution (h_o, v_o) by expressing

$$(16a) \qquad h(\sigma,t) = h_o + \lambda h^{(1)}(\omega,\tau) + \lambda^2 h^{(2)}(\omega,\tau) + \dots$$

$$(16b) \qquad v(\sigma,t) = v_o + \lambda v^{(1)}(\omega,\tau) + \lambda^2 v^{(2)}(\omega,\tau) + \dots$$

Making use of the given expansion of U/v_o with respect to λ we put (9a) and (16) into (14) and equate coefficients of like-powers of λ in the usual way. From (14a) we obtain for consecutive powers of λ:

(17a) λ^1: $[h_o v^{(1)} + (v_o - U_o)h^{(1)}]_\omega = 0$

(17b) λ^2: $[h_o v^{(2)}+(v_o-U_o)h^{(2)}]_\omega = -\left\{h^{(1)}_\tau+(h^{(1)}(v^{(1)}-U_1)]_\omega\right\}$

(17c) λ^3: $[h_o v^{(3)} + (v_o - U_o)h^{(3)}]_\omega =$

$$-\left\{h^{(2)}_\tau + [(v^{(1)} - U_1)h^{(2)}]_\omega + [h^{(1)}(v^{(2)} - U_2)]_\omega\right\}$$

and so on. From (14b) we obtain for consecutive powers of λ:

(18a) λ^o: $Kv_o{}^2 - gSh_o = 0$

(18b) λ^1: $\frac{2v^{(1)}}{v_o} - \frac{h^{(1)}}{h_o} = 0$

(18c) λ^2: $h_o(v_o - U_o)v^{(1)}_\omega + gh_o h^{(1)}_\omega$

$$+ Kv_o{}^2[(\frac{v^{(1)}}{v_o})^2 + (\frac{2v^{(2)}}{v_o} - \frac{h^{(2)}}{h_o})] = 0$$

(18d) λ^3: $h_o(v_o - U_o)v^{(2)}_\omega + gh_o h^{(2)}_\omega +$

$$Kv_o{}^2[\frac{2v^{(1)}v^{(2)}}{v_o{}^2} + (\frac{2v^{(3)}}{v_o} - \frac{h^{(3)}}{h_o})] =$$

$$-(\frac{v_o}{2} h^{(1)}_\tau - \frac{v_o U_1}{2} h^{(1)}_\omega + gh^{(1)}h^{(1)}_\omega)$$

and so on. (18a) is consistent with the constant steady flow condition (6) and (17a) and (18b) implies that $U_o/v_o = 3/2$ which is consistent with (9).

The first order system satisfied by $(h^{(1)}, v^{(1)})$ for (λ, λ^2) stretching is obtained by (17a), (17b), (18a), (18b), and (18c) after eliminating the second order

combination $(2v^{(2)}/v_o - h^{(2)}/h_o)_\omega$ between (17b) and (18c). (18b) allows a separation of $h^{(1)}$ and $v^{(1)}$ resulting in

$$(19a)\quad h^{(1)}_{\omega\omega} + A\left\{[h^{(1)}(1-h^{(1)}/h_o)]_\omega - \frac{8}{3v_o} h^{(1)}_\tau\right\} = 0$$

where A is given by (11b). For $\frac{\partial}{\partial\tau} = 0$, the first order SPW solution (12) satisfies (19a). By letting

$$(19b)\quad \nu = \frac{3v_o}{8A} = \frac{h_o(c_o^2 - v_o^2/4)}{2Kv_o}$$

which has the dimensions of kinematic viscosity and transforming $h^{(1)}$ by a simple shift factor,

$$(19c)\quad u = \frac{3}{8} v_o(1 - 2h^{(1)}/h_o)$$

(19a) becomes

$$(19d)\quad \nu u_{\omega\omega} - uu_\omega - u_\tau = 0$$

(19d) is a non-linear partial differential equation of parabolic type (like the diffusion or heat equation) and has received much attention from both mathematicians and engineers. We note with some interest that we have obtained the identical differential equation to that obtained by previous investigators approximating the Navier-Stokes and Reynold's equations. In our analysis we have started directly from bulk or hydraulic-type considerations of continuity and momentum with a turbulenceresistance term having some basis in a "mixing length theory" of turbulence. The exact significance of this observation is, of course, not clear since so many

assumptions have produced the coincident result but the connection is, at least, gratifying. Historically it appears that Bateman [10] first studied this equation in connection with the problem of a shock wave moving in a viscous compressible medium. Burgers [11] has studied this equation in his well-known contributions to the turbulence problem. Recently Hopf [12] and Cole [13] have independently given the explicit solution to the initial value problem for (19d) and $-\infty \leqq o \leqq \infty$. This is the appropriate problem for the study of flood waves and the desired results are given in two theorems by Hopf. The first theorem states that if $u_o(\omega)$ is integrable in every finite ω-interval and that

$$(20a) \quad \omega^{-2} \int_0^{\omega} u_o(\xi)d\xi \longrightarrow 0 \quad \text{as} \quad |\omega| \longrightarrow \infty$$

Then

$$(20b) \quad u(\omega,\tau) = \frac{\int_{-\infty}^{\infty} \frac{\omega-\eta}{\tau} \exp\left\{-\frac{1}{2\nu}F(\omega,\eta,\tau)\right\} d\eta}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\nu}F(\omega,\eta,\tau)\right\} d\eta}$$

where

$$(20c) \quad F(\omega,\eta,\tau) = \frac{(\omega-\eta)^2}{2\tau} + \int_0^{\eta} u_o(\xi)d\xi$$

is a regular solution of (19d) in the half plane $\tau > 0$ that satisfies the initial condition

$$(20d) \quad \int_0^{\omega} u(\xi,\tau)d\xi \longrightarrow \int_0^{a} u_o(\xi)d\xi \quad \text{as } x \longrightarrow a, \ \tau \longrightarrow 0$$

for every a. The restrictions on the initial condition $u_o(\omega)$ at $\tau = 0$ assures both uniqueness and existence

of the solution (19f) for all $\tau > 0$. In addition, Cole has considered the particular problem of an initial shock condition* for $\tau = 0$:

$$u_o(\omega) = -u_1, \text{ const. for } x < 0 \tag{20e}$$

$$= +u_1 \qquad \text{for } x > 0$$

and shows that the deviations from the SPW(12) dampen out like $\exp(-u_1^2\,\tau/4\nu)$. This initial condition and subsequent SPW at $\tau = \infty$ are shown in Fig. 2.

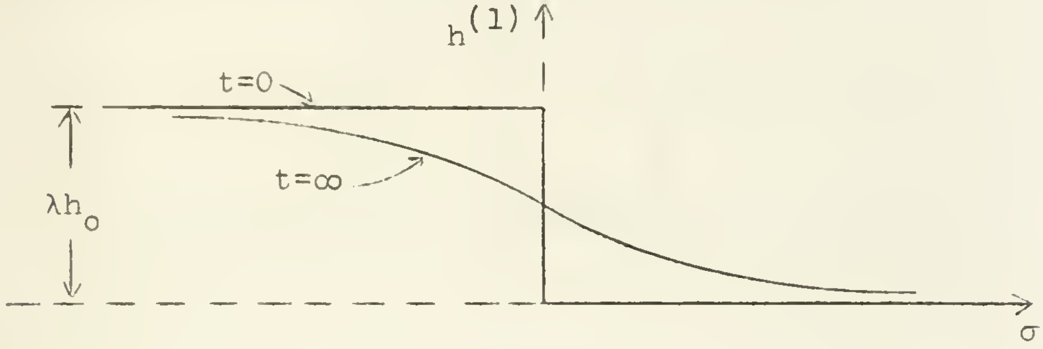


Fig. 2

Hopf's second theorem states that for initial data in which the integral

$$M = \int_{-\infty}^{\infty} u_o(\xi)\,d\xi \tag{21a}$$

exists as a sum of two improper integral $\int_0^{\infty} + \int_{-\infty}^{0}$

---

* The signs for $x \lessgtr 0$ are reversed from Cole's analysis to correspond to our physical problem of flood waves.

the solution $u(\omega,\tau)$ completely dampens out for large time like $\tau^{-1/2}$ along the line $\omega/(2\nu\tau)^{1/2}$ = const. For example, an initial condition which satisfies (21a) is

$$(21b)\qquad u_o(\omega) = + u_1,\ \text{const. for } |x| < a$$
$$= 0 \qquad\qquad \text{for } |x| > a$$

shown in Fig. 3.

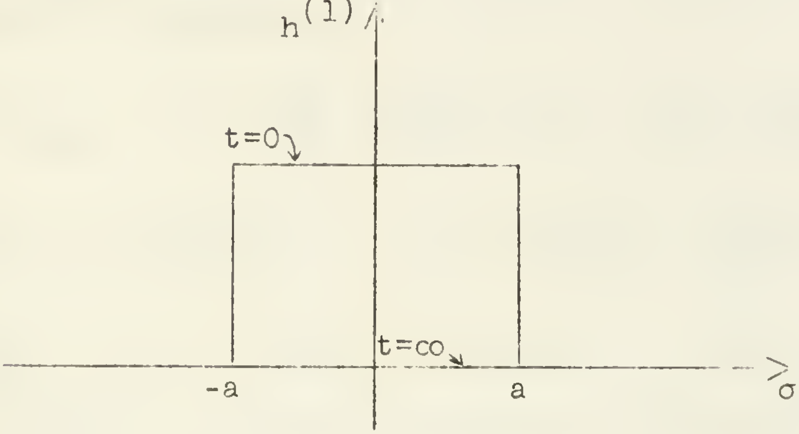
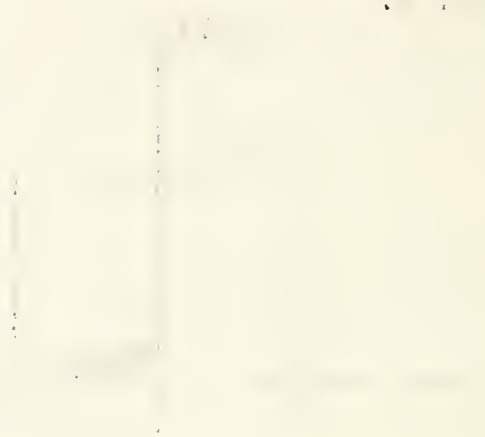


Fig. 3

The behavior of the solution as $\tau \to \infty$ for this "bump" type initial condition is consistent with the behavior of solutions for large time in Deymié's linearization.

The asymptotic nature of the stretched and perturbed system (17) and (18) and of the solutions to the corresponding initial value problem is evident by the change from the original equation (1) and (2) of hyperbolic type to appropriate equations of parabolic type. We point out also that, in this approximation procedure, the parameter λ has a definite physical meaning in contrast to the perturbation parameter of the ordinary linearization (Deymié) in which no stretching is involved; and that it is essential that the (λ,λ^2)



stretching transformation be applied in the moving (σ,t) coordinate system whereas the (λ,λ) stretching transformation described in the next section leads to the same results in either (x,t) or (σ,t) coordinate systems.

The second order system satisfied by $(h^{(2)},v^{(2)})$ for (λ,λ^2) stretching is obtained by (17) and (18) after eliminating the third order combination $(2v^{(3)}/v_o-h^{(3)}/h_o)_\omega$ between (17c) and (18d). (18c) allows a separation of $h^{(2)}$ and $v^{(2)}$ resulting in

$$(22)\quad h^{(2)}_{\omega\omega} + A\left\{ \left[\left(\frac{8U_1}{3v_o} - \frac{2h^{(1)}}{h_o}\right)h^{(2)}\right]_\omega - \frac{8}{3v_o}\,h^{(2)}_\tau \right\} =$$

$$-\left\{\frac{h_o}{2K}\left[h^{(1)}_\omega + \frac{3K}{2}\left(\frac{h^{(1)}}{h_o}\right)^2\right]_\omega + \frac{A}{6}\,h^{(1)}\left[\frac{16U_2}{v_o} + \left(\frac{h^{(1)}}{h_o}\right)^2\right]\right\}_\omega$$

where ${}^{U_1}/v_o = 3/8$ and ${}^{U_2}/v_o = -1/16$ for the SPW (cf. (9)). Unlike U_o, neither U_1 nor U_2 is fixed by the approximation procedure. (22) is a non-homogeneous linear partial differential equation of parabolic type with non-constant coefficients as functions of $h^{(1)}(\omega,\tau)$, given by the solution to the previous first order problem. For $\frac{\partial}{\partial\tau} = 0$ the second order SPW solution of the form

$$(23)\quad h^{(2)}(\omega) = \operatorname{sech}^2\frac{A\omega}{2}\int_o^{\omega}\cosh^2\frac{A\beta}{2}\sum_{n=o}^{n=3} A_n\tanh^n\frac{A\beta}{2}\,\alpha\beta$$

satisfies (22). We will not carry out a detailed analysis of the initial value problem for (22) in this report but it is to be expected that the solution will behave not too differently from that of the ordinary heat equation since the dominating terms -- second derivative term with respect to ω and first derivative term with respect to time τ of $h^{(2)}$ -- have constant coefficients.

4. Asymptotic Representation for (λ,λ) Stretching Transformation.

The approximation procedure with the (λ,λ^2) stretching transformation described in the previous section appears to be the most natural one for studying flood waves propagating downstream with the velocity of the SPW. However we shall see that the (λ,λ) stretching transformation studied in this section also leads to interesting results: 1) the analysis is somewhat simpler than that with (λ,λ^2) stretching - a consideration which may be important in practical application to flood waves, 2) a close relationship exists between (λ,λ) and (λ,λ^2) stretching, and 3) this analysis is the perturbation expansion counterpart to Hayashi's iteration procedure and has the advantage of being consistent with the SPW solution.

Thus, we consider the stretching transformation

(25) $\omega = \lambda\sigma \ , \quad \tau = \lambda t$

which transforms (2) into

(26a) $h_{\tau} + [h(v - U)]_{\omega} = 0$

(26b) $\lambda\left\{ v_{\tau} + (v - U)v_{\omega} + gh_{\omega}\right\} + \frac{Kv^2}{h} - gS = 0$

In contrast to (λ,λ^2) stretching, the (λ,λ) stretching transformation defined by (13) implies that the time and space derivative terms are of the same order of magnitude; but like (λ,λ^2) stretching, (13) still implies that all derivative terms are of lower order of magnitude as compared to non-derivative terms. As in the previous section we make perturbation expansions (16) with respect to the parameter λ of (h,v) on the constant steady flow

solution (h_o, v_o). Inserting (19a) and (16) into (26) we again equate coefficients of like-powers of λ. From (26a) we obtain for consecutive powers of λ:

$$\text{(27a)}\quad \lambda^1:\ h^{(1)}_{\tau} + [h_o v^{(1)} + (v_o - U_o)h^{(1)}]_{\omega} = 0$$

$$\text{(27b)}\quad \lambda^2:\ h^{(2)}_{\tau} + [h_o v^{(2)} + (v_o - U_o)h^{(2)}]_{\omega} + (h^{(1)}v^{(1)})_{\omega} = 0$$

$$\text{(27c)}\quad \lambda^3:\ h^{(3)}_{\tau} + [h_o v^{(3)} + (v_o - U_o)h^{(3)}]_{\omega} + [(v^{(1)} - U_1)h^{(2)} + h^{(1)}(v^{(2)} - U_2)]_{\omega} = 0$$

and so on. From (26b) we obtain for consecutive powers of λ:

$$\text{(28a)}\quad \lambda^o:\ Kv_o^{\,2} - gSh_o = 0$$

$$\text{(28b)}\quad \lambda^1:\ \left(\frac{2v^{(1)}}{v_o} - \frac{h^{(1)}}{h_o}\right) = 0$$

$$\text{(28c)}\quad \lambda^2:\ v^{(1)}_{\tau} + (v_o - U_o)v^{(1)}_{\omega} + gh^{(1)}_{\omega} + \frac{Kv_o^{\,2}}{h_o}\left[\left(\frac{v^{(1)}}{v_o}\right)^2 + \left(\frac{2v^{(2)}}{v_o} - \frac{h^{(2)}}{h_o}\right)\right] = 0$$

$$\text{(28d)}\quad \lambda^3:\ v^{(2)}_{\tau} + (v_o - U_o)v^{(2)}_{\omega} + gh^{(2)}_{\omega} + \frac{1}{h_o}\left[h^{(1)}v^{(1)}_{\tau} + h_o(v^{(1)} - U_1)v^{(1)}_{\omega} + (v_o - U_o)h^{(1)}v^{(1)}_{\omega} + gh^{(1)}h^{(1)}_{\omega}\right] + \frac{Kv_o^{\,2}}{h_o}\left(\frac{2v^{(1)}v^{(2)}}{v_o^{\,2}} + \frac{2v^{(3)}}{v_o} - \frac{h^{(3)}}{h_o}\right) = 0$$

and so on. (28a) is consistent with the constant steady flow condition (6).

The first order system satisfied by $(h^{(1)}, v^{(1)})$ for (λ, λ) stretching is obtained by (27a) and (28b) which allows a separation of $h^{(1)}$ and $v^{(1)}$ resulting in

$$(29a) \qquad h^{(1)}_{\tau} + (\tfrac{3}{2} v_o - U_o) h^{(1)}_{\omega} = 0$$

If the propagation velocity U is taken to be that for the SPW then $U_o/v_o = 3/2$ and (29a) becomes simply

$$(29b) \qquad h^{(1)}_{\tau} = 0$$

(29b) states that all first order disturbances remain unchanged and propagate with the velocity of the SPW. We note that, in contrast to (λ, λ^2) stretching, (λ, λ) stretching does not fix the value of U_o although the coefficient $(\frac{3}{2} v_o - U_o)$ of $h^{(1)}_{\omega}$ in (29a) implies the importance of the particular value $U_o/v_o = 3/2$. Also, if the stretching transformation and the perturbation expansions are made in the (x,t) coordinate system as given by (1) then the first order equation for (λ, λ) stretching is

$$(30) \qquad h^{(1)}_{\tau} + \tfrac{3}{2} v_o h^{(1)}_{\omega} = 0$$

which is identical to (29b) under the coordinate transformation $(x,t) \longrightarrow (\sigma, t)$ where $\sigma = x - Ut$ and $U = U_o = 3v_o/2$ is the first order approximation to the propagation velocity of the SPW. Thus the following higher order systems are given for the special case of U given by (9b) for the SPW.

The second order system satisfied by $(h^{(2)}, v^{(2)})$ for (λ,λ) stretching is obtained by (27a), (27b), (28b) and (28c) which allows a separation of $h^{(2)}$ and $v^{(2)}$ resulting in

$$h^{(2)}_{\tau} = \nu[h^{(1)}_{\omega} + Ah^{(1)}(1 - h^{(1)}/h_o)]_{\omega} \tag{30-}$$

where A is defined by (11b) and ν by (19b). The right hand side of (30) is nullified by the first order SPW (cf. (19a) for $\frac{\partial}{\partial\tau} = 0$) and implies $h^{(2)}_{\tau} = 0$ for this special case. This is consistent with the original system (2) and is, in this sense, an improvement over Hayashi's iteration procedure. However, any first order solution which does not nullify the right hand side of (30) implies that $h^{(2)}$ increases linearly with time and thus becomes unbounded as $t \longrightarrow \infty$. This implies that the (λ,λ) stretching transformation inherently leads to solutions singular at $t = \infty$ for almost all types of disturbances whether a perturbation expansion or an iteration (Hayashi) is made.

The third order system satisfied by $(h^{(3)}, v^{(3)})$ for (λ,λ) stretching is obtained by (27) and (28) and (28c) allows a separation of $h^{(3)}$ and $v^{(3)}$ resulting in

$$h^{(3)} = \nu\Big\{h^{(2)} + A(1 - \frac{2h^{(1)}}{h_o})h^{(2)} + \frac{h_o}{2K}[h^{(1)}_{\omega} + \frac{3K}{2}(\frac{h^{(1)}}{h_o})^2]_{\omega} - \frac{A}{6} h^{(1)}[1 - (\frac{h^{(1)}}{h_o})^2]\Big\}_{\omega} \tag{31}$$

As expected from the study of the second order system, i.e., equation (30), the right hand side of (31) is nullified by the SPW solution (cf. (23) for $\frac{\partial}{\partial\tau} = 0$).

This result is again consistent with the original system (2).

If we now make a critical comparison of the results of our analysis by (λ,λ) and (λ,λ^2) stretching, our preference from the viewpoint of the theory of flood waves is decidedly in favor of (λ,λ^2) stretching. The first order system for (λ,λ^2) stretching is time-dependent and solutions damp out as $t \to \infty$. For (λ,λ) stretching we must investigate the second order system in order to get time-dependence of solutions and these solutions are unbounded for large time.

Remarks

In our study of the theory of flood waves which propagate downstream with the velocity of the steady progressing wave we find that a natural way of obtaining appropriate asymptotic representations is by stretching transformations on both the space and time variables combined with the usual perturbation expansion but with respect to the stretching parameter. There appears to be a best stretching transformation for this flood routing problem, namely, the (λ, λ^2) stretching, which must be performed in the appropriate coordinate system ((σ, t) instead of (x, t)). However, there may be other stretching transformations, e.g., (λ, λ) stretching, which yield additional asymptotic representations giving valuable information and which may be particularly useful in application to other physical problems.

Finally, we note that this investigation has been made for the particular purpose of obtaining reasonable asymptotic representations of relevant physical phenomena. We have used the term "asymptotic" without supplying the estimates that are implied in the mathematical usage of the word "asymptotic". We only hope further work will be stimulated in this direction.

References

1. H. Rouse, *Engineering Hydraulics*, Wiley, New York and London, 1950. (Chapter 10, *Flood Routing*, pp.635-710, by B. R. Gilchrest).
2. J. J. Stoker, *Numerical Solution of Flood Prediction and River Regulation Problems*, *Report I: Derivation of Basic Theory and Formulation of Numerical Methods of Attack*, New York University, Institute of Math. Sc. Report No. IMM-NYU-200, (October, 1953).
3. E. Isaacson, J. J. Stoker and A. Troesch, *Numerical Solution of Flood Prediction and River Regulation Problems*, *Report II: Numerical Solution of Flood Problems in Simplified Models of the Ohio River and the Junction of the Ohio and Mississippi Rivers*. New York University, Institute of Math. Sc. Report No. IMM-NYU-205, (January, 1954).
4. K. O. Friedrichs, "*On the Derivation of the Shallow Water Theory*". Appendix to *The Formation of Breakers and Bores* by J. J. Stoker, Communications on Applied Mathematics and Mechanics, New York University, Vol. 1, No. 1 (January, 1948).
5. J. B. Keller, *The Solitary Wave and Periodic Waves in Shallow Water*, Communications on Applied Mathematics and Mechanics, New York University, Vol. 1, No. 4 (December, 1948).
6. L. Prandtl, *Essentials of Fluid Dynamics*, Hafner, New York, pp. 161-2, 1952.
7. J. J. Stoker, *The Formation of Breakers and Bores*, Communications on Applied Mathematics and Mechanics, New York University, Vol. 1, No. 1, (January, 1948).
8. Ph. Deymié, *Propagation d'une intumescence allongée (problème aval)*, Fifth Int. Congress for Applied Mech., Proceedings, Wiley, New York, September, 1938.

9. T. Hayashi, *Mathematical Theory of Flood Waves*, Proc. of the First Japan. National Congress for Applied Mech., 1951.
10. H. Bateman, *Some Recent Researches on the Motion of Fluids*, Monthly Weather Review, Vol. 43, pp. 163-170, (1915).
11. J. Burgers, *A Mathematical Model Illustrating the Theory of Turbulence*, Advances in Applied Mechanics, Vol. 1, Academic Press, New York, pp. 171-199, 1948.
12. E. Hopf, *The Partial Differential Equation*, $u_t + uu_x = \mu u_{xx}$, Communications on Pure and Applied Mathematics, Vol. III, No. 3, (September, 1950).
13. J. D. Cole, On a Quasi-Linear Parabolic Equation *Occurring in Aerodynamics*, Quarterly of Applied Math., Vol. IX, No. 3, (October, 1951).

Date Due

PRINTED IN U. S. A.

NYU-IMM-210

Morikawa

NYU-IMM-210

Morikawa

AUTHOR

On the theory of

TITLE

flood waves in rivers

NYU-IMM-210

Morikawa, George K

On the theory of flood waves in rivers.

Manufactured in the United States for New York University Press
by the University's Office of Publications and Printing